42 YEARS OF EXCELLENCE

SRAMAYEVA JAYATHE

NARAYANA EDUCATIONAL INSTITUTIONS

NARAYANA'S SENSATIONAL SUCCESS ACROSS INDIA

7 Students Secured 100 Percentile in All India JEE Main-2020

BELOW 10
21 RANKS
All Cat

BELOW 100
113 RANKS
All Cat

100 PERCENTILE	100 PERCENTILE	100 PERCENTILE	100 PERCENTILE	100 PERCENTILE	100 PERCENTILE	100 PERCENTILE
M LIKHITH REDDY HT No. TL0305724	SHIVA KRISHNA HT No. TL01330751	TANUJA HT No. TL01327005	SWAYAM CHUBE HT No. MR16303069	HARSHVARDAHAN AGARWAL HT No. HR09300789	YESHASH CHANDRA HT No. TL02305081	PARTH DWIVEDI Appl. No. 200310166791

ADMISSIONS OPEN (2020-21)

OUR REGULAR CLASSROOM PROGRAMME

One Year Classroom Program
JEE/NEET-2021
(for students moving from XI to XII)

Two Year Classroom Program
JEE/NEET-2022
(for students moving from X to XI)

Three Year Integrated Classroom Program
JEE/NEET-2023
(for students moving from IX to X)

Four Year Integrated Classroom Program
JEE/NEET-2024
(for students moving from VIII to IX)

FOUNDATION PROGRAMMES
For NTSE, NSEJS, JSTSE, Olympiads & School/Board Exams
(for students moving to Class VI, VII, VIII, IX & X)

APEX BATCH
Two years school Integrated Classroom Program - 2022
For JEE Main & Advance / NEET (for XI Studying Students)
Course Feature: Complete Coverage of CBSE ▪ Regular Classes ▪ Weekly Test & Regular Analysis ▪ Lab Facility ▪ Motivation & Counseling ▪ Competetive Exam Prep. ▪ Ample time for self stuty

Online Classes for IIT/NEET/Foundation/Olympiads

- Access Recording of Past Classes on n-Learn App
- Online Parent Teacher Meeting
- Personalized Extra Classes & Live Doubt Solving
- Hybrid/Customized Classroom model
- Video Solution of Weekly/Fortnightly Test
- Printed Study Material will be sent by us
- n-Lean App
- Counselling Motivational sessions
- Affordable Fee
- Doubt Classes / Practice Classes
- Provision to Convert from online to regular classroom programme
- Once Classes resume by just paying nominal fee

Online Test

- Micro & Macro Analysis
- Relative performance (All India Ranking)
- Question wise Analysis
- Unlimited Practice Test
- Grand Test

NARAYANA Digital Classes
STUDY ONLINE FROM HOME

For Class 7th to 12th+

TOLL FREE : 1800 102 3344 | Visit us at : www.narayanagroup.com

JEE-MAIN-2021

FEBRUARY ATTEMPT

26.02.2021_SHIFT-II

MATHEMATICS

MATHEMATICS

1. If $0 < a, b < 1$, $\tan^{-1}a + \tan^{-1}b = \frac{\pi}{4}$ find value of $(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right)$ ….. ∞

(1) $\ell n\ 3$ (2) $2\ \ell n\ 2$ (3) $\ell n\ 2$ (4) $\ell n\ 2 + \ell n\ 3$

Ans. (3)

Sol. $\tan^{-1}\left(\frac{a+b}{1-ab}\right) = \frac{\pi}{4}$ $\Rightarrow a + b = 1 - ab$ $\Rightarrow (1+a)(1+b) = 2$

Now, $a + b - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) \ldots\ldots \infty$

$= \left(a - \frac{a^2}{2} + \frac{a^3}{3} \ldots\ldots\right) + \left(b - \frac{b^2}{2} + \frac{b^3}{3} \ldots\ldots\right)$

$= \ell n(1+a) + \ell n(1+b) = \ell n(1+a)(1+b) = \ell n\ 2$

2. If $f(x) = \int_0^x e^t f(t)\,dt + e^x$, then f(x) is equal to

(1) $2e^{e^x-1}$ (2) $2e^{e^x-1}+1$ (3) $e^{e^x-1}+1$ (4) $2e^{e^x-1}-1$

Ans. (4)

Sol. $f'(x) = e^x \cdot f(x) + e^x$

$\Rightarrow \frac{f'(x)}{f(x)+1} = e^x$ $\Rightarrow \ell n(f(x)+1) = e^x + c$

put x = 0

$\ell n\ 2 = 1 + c$

$\therefore \ell n(f(x)+1) = e^x + \ell n\ 2 - 1$

$\Rightarrow f(x) + 1 = 2 \cdot e^{e^x-1}$ $\Rightarrow f(x) = 2e^{e^x-1} - 1$

3. A seven digit number has been formed by using digit 3, 3, 4, 4, 4, 1, 1 (by taking all at a time). Probability that number is even.

(1) $\frac{2}{7}$ (2) $\frac{3}{7}$ (3) $\frac{5}{14}$ (4) $\frac{3}{14}$

Ans. (2)

Sol. $n(S) = \frac{7!}{2!3!2!}$

$n(E) = \frac{6!}{2!2!2!}$

$P(E) = \frac{n(E)}{n(S)} = \frac{6!}{7!} \times \frac{2!3!2!}{2!2!2!}$

$= \frac{1}{7} \times 3 = \frac{3}{7}$

4. If A_1 & A_2 are area bounded by :

A_1 : y = sin x, y = cos x & y-axis in I^{st} quadrant

A_2 : y = sin x, y = cos x & x-axis & x = $\frac{\pi}{2}$.

Then

(1) $A_1 : A_2 = 1 : \sqrt{2}$; $A_1 + A_2 = 1$

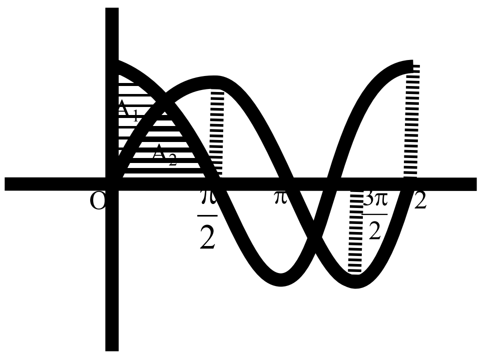
(2) $A_1 : A_2 = \sqrt{2} : 1$; $A_1 + A_2 = \sqrt{2} + 1$

(3) $A_1 : A_2 = 1 : 2$; $A_1 + A_2 = 2$

(4) $A_1 : A_2 = 1 : 2$; $A_1 + A_2 = 1$

Ans. (1)

Sol. $A_1 + A_2 = \int_0^{\pi/2} \cos x \, . \, dx = \sin x\Big|_0^{\pi/2} = 1$



$A_1 = \int_0^{\pi/4} (\cos x - \sin x)\, dx = (\sin x + \cos x)\Big|_0^{\pi/4} = \sqrt{2} - 1$

$\therefore \; A_2 = 1 - (\sqrt{2} - 1) = 2 - \sqrt{2}$

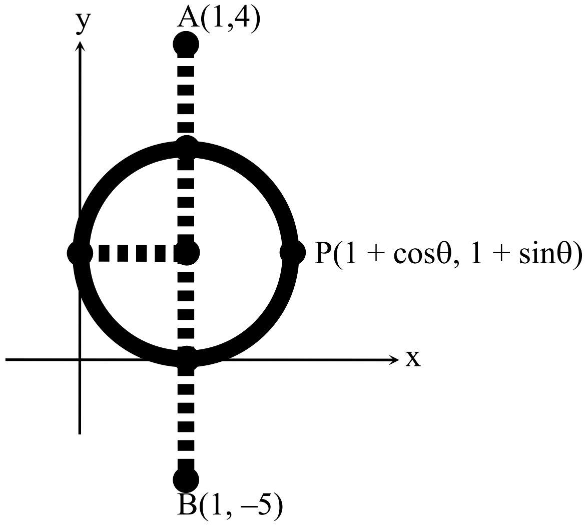
$\therefore \; \frac{A_1}{A_2} = \frac{\sqrt{2}-1}{\sqrt{2}(\sqrt{2}-1)} = 1 : \sqrt{2}$

5. Consider the circle $(x - 1)^2 + (y - 1)^2 = 1$; A(1, 4) B(1, –5). If P is a point on circle. Such that (PA) + (PB) is maximum, then P, A, B lie on ?

(1) ellipse (2) hyperbola (3) Straight line (4) None of these

Ans. (3)

Sol.



$\therefore \; PA^2 = \cos^2\theta + (\sin\theta - 3)^2 = 10 - 6\sin\theta$

$PB^2 = \cos^2\theta + (\sin\theta - 6)^2 = 37 - 12\sin\theta$

$PA^2 + PB^2 = 47 - 18\sin\theta|_{max.} \Rightarrow \theta = \frac{3\pi}{2}$

$\therefore$ P, A, B lie on a line x = 1

6. If A triangle is inscribed in a circle of radius r, then which of the following triangle can have maximum area

(1) equilateral triangle with height $\frac{2r}{3}$

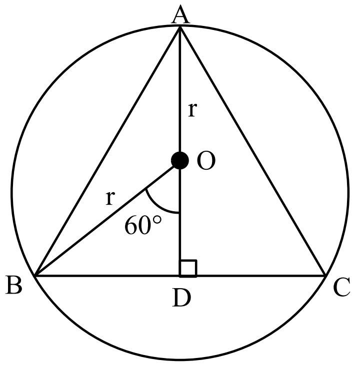
(2) right angle triangle with side 2r, r

(3) equilateral triangle with side $\sqrt{3}r$

(4) isosceles triangle with base 2r

Ans. (3)

Sol.



$OD = r\cos 60° = \frac{r}{2}$

$\text{Height} = AD = \frac{3r}{2}$

$\text{Now } \sin 60° = \frac{3\frac{r}{2}}{AB}$

$\Rightarrow AB = \sqrt{3}r$

7. If f'(a) = 2 and f(a) = 4, then find value of $\lim_{x\to a}\frac{xf(a)-af(x)}{x-a}$

(1) 4 – 2a (2) 2a – 4 (3) 0 (4) a – 4

Ans. (1)

Sol. By L–H rule

$L = \lim_{x\to a}\frac{f(a)-af'(x)}{1}$

$\therefore L = 4 - a(2)$

8. If $\vec{a}_1 = x\hat{i}-\hat{j}+\hat{k}$, $\vec{a}_2 = \hat{i}+y\hat{j}+z\hat{k}$ are collinear vector, then a unit vector which is parallel to $x\hat{i}+y\hat{j}+z\hat{k}$, can be

(1) $\frac{1}{\sqrt{3}}(\hat{i}-\hat{j}+\hat{k})$ (2) $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ (3) $\frac{1}{\sqrt{2}}(\hat{j}-\hat{k})$ (4) $\frac{\hat{i}-\hat{j}-\hat{k}}{\sqrt{3}}$

Ans. (1)

Sol. $\frac{x}{1} = -\frac{1}{y} = \frac{1}{z} = \lambda(\text{let})$

Unit vector parallel to $x\hat{i}+y\hat{j}+z\hat{k} = \pm \frac{\left(\lambda\hat{i}-\frac{1}{\lambda}\hat{j}+\frac{1}{\lambda}\hat{k}\right)}{\sqrt{\lambda^2+\frac{2}{\lambda^2}}}$

for $\lambda = 1$ it is $\pm \frac{(\hat{i}-\hat{j}+\hat{k})}{\sqrt{3}}$

9. If $y + z = 5$, $\frac{1}{y}+\frac{1}{z}=\frac{5}{6}, y>z$

If prime factorization of a natural number $N = 2^x\ 3^y\ 5^z$. Find number of odd divisors of N including 1.

(1) 6x (2) 6 (3) 12 (4) 11

Ans. (3)

Sol. Solving given two equation we get y = 3, z = 2

$\Rightarrow N = 2^x\ 3^3\ 5^2$

number of odd divisor = (2 + 1) (3 + 1) = 12

10. If a curve y = f(x) in given by $\frac{dy}{dx}=\frac{xy^2+y}{x}$ passing through (–2, 3) meets the line L = 0 at (3, y) then y is

(1) $\frac{-11}{19}$ (2) $\frac{-18}{19}$ (3) $\frac{-11}{29}$ (4) $\frac{11}{19}$

Ans. (2)

Sol. $\frac{dy}{dx}=\frac{xy^2+y}{x}$

$\Rightarrow \frac{xdy-ydx}{y^2} = x\,dx$

$\Rightarrow -d\left(\frac{x}{y}\right) = d\left(\frac{x^2}{2}\right)$

$\Rightarrow \frac{-x}{y} = \frac{x^2}{2} + C$

It passes through (–2, 3)

$\Rightarrow \frac{2}{3} = 2 + C$

$\Rightarrow C = \frac{-4}{3}$

$\therefore$ curve is $\frac{-x}{y} = \frac{x^2}{2} - \frac{4}{3}$

It also passes through (3, y)

$\frac{-3}{y} = \frac{9}{2} - \frac{4}{3}$

$\Rightarrow \frac{-3}{y} = \frac{19}{6}$

$\Rightarrow y = -\frac{18}{19}$

11. If $f(x) = \int_1^x \frac{\log_e(t)}{(1+t)} dt$, then $f(e) + f\left(\frac{1}{e}\right)$ is

(1) 0 (2) 1 (3) –1 (4) $\frac{1}{2}$

Ans. (4)

Sol. $f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\ell nt}{1+t} dt + \int_1^{1/e} \frac{\ell nt}{1+t} dt = I_1 + I_2$

$I_2 = \int_1^{1/e} \frac{\ell nt}{1+t} dt$ put $t = \frac{1}{z}$ $dt = -\frac{dz}{z^2}$

$= \int_1^e -\frac{\ell nz}{1+\frac{1}{z}} \times \left(-\frac{dz}{z^2}\right) = \int_1^e \frac{\ell nz}{z(z+1)} dz$

$f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\ell nt}{1+t} dt + \int_1^e \frac{\ell nt}{t(t+1)} dt = \int_1^e \frac{\ell nt}{1+t} + \frac{\ell nt}{t(t+1)} dt$

$= \int_1^e \frac{\ell nt}{t} dt = \int_0^1 u\, du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1}{2}$

12. Consider the system of equation $x + 2y - 3z = a$, $2x + 6y - 11z = b$, $x - 2y + 7z = c$ then

(1) unique solution for $\forall$ a, b, c
(2) infinite solution for $5a = 2b + c$
(3) no solution for all a, b, c
(4) unique solution for $5a = 2b + c$

Ans. (2)

Sol. $D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{vmatrix}$

$= 20 - 2(25) - 3(-10)$

$= 20 - 50 + 30 = 0$

$D_1 = \begin{vmatrix} a & 2 & -3 \\ b & 6 & -11 \\ c & -2 & 7 \end{vmatrix}$

$= 20a - 2(7b + 11c) - 3(-2b - 6c)$

$= 20a - 14b - 22c + 6b + 18c$

$= 20a - 8b - 4c$

$= 4(5a - 2b - c)$

$D_2 = \begin{vmatrix} 1 & a & -3 \\ 2 & b & -11 \\ 1 & c & 7 \end{vmatrix}$

$= 7b + 11c - a(25) - 3(2c - b)$

$= 7b + 11c - 25a - 6c + 3b$

$= -25a + 10b + 5c$

$= -5(5a - 2b - c)$

$D_3 = \begin{vmatrix} 1 & 2 & a \\ 2 & 6 & b \\ 1 & -2 & c \end{vmatrix}$

$= 6c + 2b - 2(2c - b) - 10a$

$= -10a + 4b + 2c$

$= -2(5a - 2b - c)$

for infinite solution

$D = D_1 = D_2 = D_3 = 0$

$\Rightarrow 5a = 2b + c$

13. A function f(k) is defined A to A where A = {1,2,3,4,5...........10}, such that

$$f(k) = \begin{cases} k+1, & k \in \text{odd} \\ k, & k \in \text{even} \end{cases}.$$

If gof(x) = f(x) then number of mapping of g(x) from $A \to A$ is

(1) ${}^{10}C_5$
(2) 10^5
(3) 5^5
(4) 5!

Ans. (2)

Sol. $g(f(x)) = f(x)$

$\Rightarrow g(x) = x$, when x is even.

$\therefore$ So total number of functions from A to A

$= 10^5 \times 1 = 10^5$

14. If F_1 (A, B, C) = $(\sim A \vee B) \vee (\sim A) \vee (\sim C \wedge (A \vee B))$

F_2 = (A, B, C) = $(A \vee B) \vee (A \rightarrow \sim B)$

Then which of the following is true :

(1) F_1 is Tautology and F_2 is Tautology

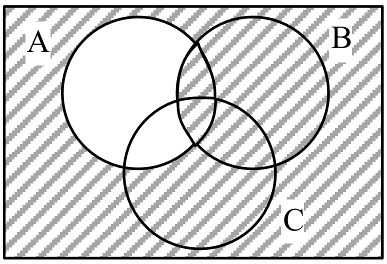
(2) F_1 is Tautology and F_2 is not Tautology

(3) F_1 is not Tautology and F_2 is Tautology

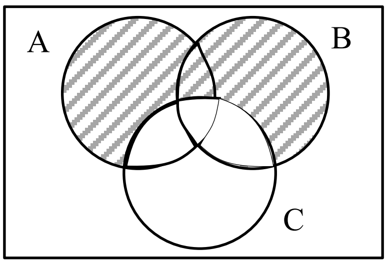
(4) Neither is Tautology

Ans. (3)

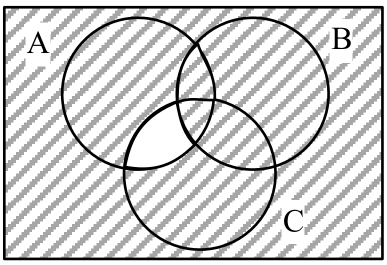
Sol. $(\sim A \vee B) \equiv$



$\sim C \wedge (A \vee B)$

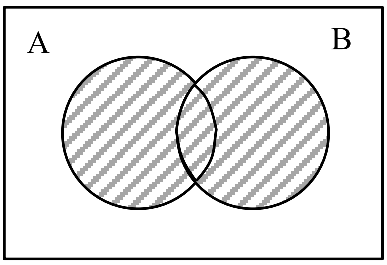


$\therefore F_1$:

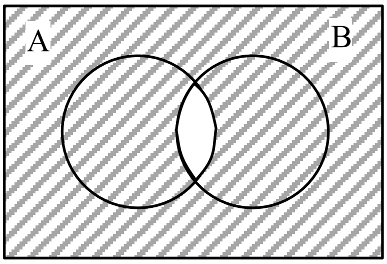


$\Rightarrow$ Not tautology

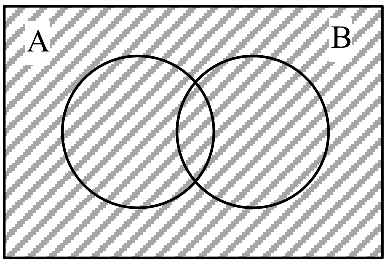
$A \vee B \equiv$



$A \to \sim B$



$\therefore F_2 \to$



Tautology

Truth table for F_1 (A, B, C)

A	B	C	~A	~C	A∨B	~A∨B	~C∧(A∨B)	(~A∨B)∨(~C∧(A∨B))∨~A
T	T	T	F	F	T	T	F	T
T	F	F	F	T	T	F	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	F	T	F	F	F
F	T	T	T	F	T	T	F	T
F	F	F	T	T	F	T	F	T
F	T	F	T	T	T	T	T	T
F	F	T	T	F	F	T	F	T

Truth table for F_2

A	B	A∨B	~B	A→~B	(A∨B)∨(A→~B)
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	T	T
F	F	F	T	T	T

F_1 not shows tautology and F_2 shows tautology.

15. $\sum_{n=1}^{\infty} \frac{n^2+6n+10}{(2n+1)!} =$

(1) $\frac{41e}{8}+\frac{19e}{8}-10$

(2) $\frac{41e}{8}+\frac{19}{e}-10$

(3) $\frac{41}{8}e-\frac{19}{8e}-10$

(4) $\frac{41}{8}e-\frac{19}{8e}-80$

Ans. (3)

Sol. $\sum_{n=1}^{\infty}\frac{n^2+6n+10}{(2n+1)!}$

put 2n + 1 = r, where r = 3,5,7,.........

$\Rightarrow n = \frac{r-1}{2}$

$$\frac{n^2-6n+10}{(2n+1)!}=\frac{\left(\frac{r-1}{2}\right)^2+3r-3+10}{r!}=\frac{r^2+10r+29}{4r!}$$

Now $\sum_{r=3,5,7}\frac{r(r-1)+11r+29}{4r!}=\frac{1}{4}\sum_{r=3,5,7,........}\left(\frac{1}{(r-2)!}+\frac{11}{(r-1)!}+\frac{29}{r!}\right)$

$$=\frac{1}{4}\left\{\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+........\right)+11\left(\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+........\right)+29\left(\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+........\right)\right\}$$

$$=\frac{1}{4}\left\{\frac{e-\frac{1}{e}}{2}+11\left(\frac{e+\frac{1}{e}-2}{2}\right)+29\left(\frac{e-\frac{1}{e}-2}{2}\right)\right\}$$

$$=\frac{1}{8}\left\{e-\frac{1}{e}+11e+\frac{11}{e}-22+29e-\frac{29}{e}-58\right\}$$

$$=\frac{1}{8}\left\{41e-\frac{19}{e}-80\right\}$$

16. Foot of the perpendicular from the points (3, 4, 1) on the line of intersection of the planes x + 2y + z – 6 = 0 & y + 2z = 4 is

(1) $\left(\frac{10}{7},\frac{12}{7},\frac{8}{7}\right)$ (2) $\left(\frac{10}{7},\frac{-12}{7},\frac{8}{7}\right)$ (3) $\left(\frac{10}{7},\frac{-12}{7},\frac{-8}{7}\right)$ (4) $\left(\frac{-10}{7},\frac{12}{7},\frac{8}{7}\right)$

Ans. (1)

Sol. Let D.R's of line are a, b, c

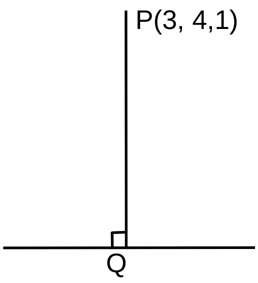
$\therefore$ a + 2b + c = 0

0.a + b + 2c = 0

$\frac{a}{3}=\frac{b}{-2}=\frac{c}{1}$

Points on the line is (–2, 4, 0)

$\therefore$ equation of line is $\frac{x+2}{3}=\frac{y-4}{-2}=\frac{z}{1}=1$



Point Q on the line is $(3\lambda - 2, -2\lambda + 4, \lambda)$

DR's of PQ ; $3\lambda - 5, -2\lambda, \lambda - 1$

DR's of y lines are 3, –2, 1

Since PQ ⊥ line $\Rightarrow 3(3\lambda - 5) - 2(-2\lambda) + 1(\lambda - 1) = 0$

$\Rightarrow 14\lambda - 16 \Rightarrow \lambda = \frac{8}{7}$

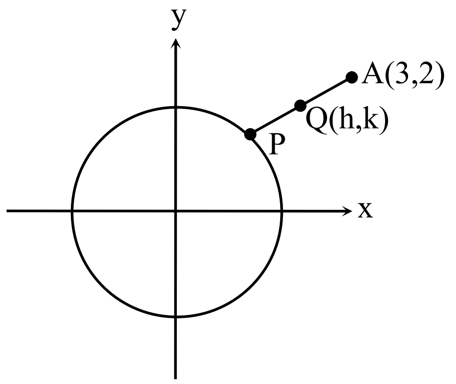
$\therefore \; Q\left(\frac{10}{7}, \frac{12}{7}, \frac{8}{7}\right)$

17. From the point A(3, 2), a line is drawn to any point on the circle $x^2 + y^2 = 1$. if locus of midpoint of this line segment is a circle, the its radius is

(1) $\frac{\sqrt{13}}{2}$ (2) $\frac{1}{2}$ (3) $\frac{\sqrt{11}}{2}$ (4) $\sqrt{11}$

Ans. (1)

Sol.



$\therefore P \equiv (2h - 3, 2k - 2) \rightarrow$ on circle

$\therefore \left(h - \frac{3}{2}\right)^2 + (k - 1)^2 = \frac{1}{4}$

$\Rightarrow$ radius $= \frac{1}{2}$

18. Given $f(x) = \sin^{-1}x$, $g(x) = \frac{x^2 - x - 2}{2x^2 - x - 6}$, $x \neq 2$ and $g(2) = \lim_{x \to 2} g(x)$ find domain fog(x)

(1) $(-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right)$ (2) $(-\infty, -1] \cup [2, \infty)$

(3) $\left[-2, -\frac{4}{3}\right]$ (4) $(-\infty, 2)$

Ans. (1)

Sol. $g(2) = \lim_{x\to 2} \frac{(x-2)(x+1)}{(2x+3)(x-2)} = \frac{3}{7}$

For domain of fog (x)

$\left|\frac{x^2-x-2}{2x^2-x-6}\right| \le 1 \quad \Rightarrow \quad (x+1)^2 \le (2x+3)^2$

$\Rightarrow (3x+4)(x+2) \ge 0$

$x \in (-\infty, -2] \cup \left(-\frac{4}{3}, \infty\right]$

19. If $f(x) = \begin{cases} 2\sin\left(-\frac{\pi x}{2}\right) & ; \quad x < -1 \\ |ax^2 + x + b| & ; \quad -1 \le x < 1 \\ \sin(\pi x) & ; \quad 1 \le x \end{cases}$ is continuous $\forall\ x \in R$, then find (a + b)

(1) –1 (2) 1 (3) 2 (4) –2

Ans. (1)

Sol. If f is continuous at x = –1, then

$f(-1^-) = f(-1)$

$\Rightarrow 2 = |a - 1 + b|$

$\Rightarrow |a + b - 1| = 2 \quad$ ..... (i)

similarly

$f(1^-) = f(1)$

$\Rightarrow |a + b + 1| = 0$

$\Rightarrow a + b = -1$

20. If $I_{m,n} = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$, m, n ≥ 1 and $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} = \alpha I_{m,n}$, then find 'α'

(1) 1 (2) 2 (3) 0 (4) –1

Ans. (1)

Sol. $I_{m,n} = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$ put $x = \frac{1}{y+1}$

$I_{m,n} = \int_\infty^0 \frac{y^{n-1}}{(y+1)^{m+n}}(-1)\,dy = \int_0^\infty \frac{y^{n-1}}{(y+1)^{m+n}}\,dy \quad$ …..(i)

Similarly $I_{m,n} = \int_0^1 x^{n-1}.(1-x)^{m-1}\,dx$

$\Rightarrow I_{m,n} = \int_0^\infty \frac{y^{m-1}}{(y+1)^{m+n}}\,dy$ .....(ii)

From (i) & (ii)

$2I_{m,n} = \int_0^\infty \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}\,dy$

$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}\,dy + \int_1^\infty \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}\,dy$

Put $y = \frac{1}{z}$

$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}\,dy + \int_1^\infty \frac{z^{m-1}+z^{n-1}}{(z+1)^{m+n}}\,dz$

$\Rightarrow I_{m,n} = \int_0^1 \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+1}}\,dy \Rightarrow \alpha = 1$

21. If slope of common tangent to curve $4x^2 + 9y^2 = 36$ and $4x^2 + 4y^2 = 31$ is m then m^2 is equal to

Ans. 3

Sol. E : $\frac{x^2}{9}+\frac{y^2}{4} = 1$ C : $x^2 + y^2 = \frac{31}{4}$

equation of tangent to ellipse

$y = mx \pm \sqrt{9m^2+4}$ ....(i)

equation of tangent to circle

$y = mx \pm \sqrt{\frac{31}{4}m^2+\frac{31}{4}}$ ...(ii)

Comparing equation (i) & (ii)

$9m^2 + 4 = \frac{31m^2}{4}+\frac{31}{4}$

$\Rightarrow 36m^2 + 16 = 31m^2 + 31$

$\Rightarrow 5m^2 = 15$

$\Rightarrow m^2 = 3$

22. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ and $A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then find $(\alpha - \beta)$

Ans. 4

Sol. $$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

.

.

.

.

.

$$A^{19} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{19} & 0 \\ 3 & 0 & -1 \end{bmatrix}, A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{20} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{L.H.S} = A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1+\alpha+\beta & 0 & 0 \\ 0 & 2^{20}+\alpha 2^{19}+2\beta & 0 \\ 3\alpha+3\beta & 0 & 1-\alpha-\beta \end{bmatrix}$$

$$\text{R.H.S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \alpha + \beta = 0 \text{ and}$$

$$2^{20} + \alpha 2^{19} + 2\beta = 4$$

$$2^{20} + \alpha(2^{19} - 2) = 4$$

$$2 = \frac{4 - 2^{20}}{2^{19} - 2} = -2$$

$\beta = 2 \Rightarrow (\alpha - \beta) = 4$

23. In normal drawn to the curve at any point passes through a fixed point (a, b). The curve passes through (3, –3) & $(4, -2\sqrt{2})$ such that $a - 2\sqrt{2}\,b = 3$. Find $a^2 + b^2 + ab$

Ans. 9

Sol. Let the equation of normal is $Y - y = -\frac{1}{m}(X - x)$

Satisfy (a, b) in it $b - y = -\frac{1}{m}(a - x)$

$\Rightarrow (b - y)\, dy = (x - a)\, dx$

$by - \frac{y^2}{2} = \frac{x^2}{2} - ax + c$ ……(i)

It passes through (3, –3) & $(4, -2\sqrt{2})$

$\therefore\ -3b - \frac{9}{2} = \frac{9}{2} - 3a + c$

$\Rightarrow -6b - 9 = 9 - 6a + 2c$

$\Rightarrow 6a - 6b - 2c = 18$

$\Rightarrow 3a - 3b - c = 9$ ……(ii)

Also

$-2\sqrt{2}\,b - 4 = 8 - 4a + c$

$4a - 2\sqrt{2}\,b - c = 12$ ……(iii)

Also $a - 2\sqrt{2}\,b = 3$ ……(iv) (given)

(ii) – (iii) $\Rightarrow -a + (2\sqrt{2} - 3)\,b = -3$ ……(v)

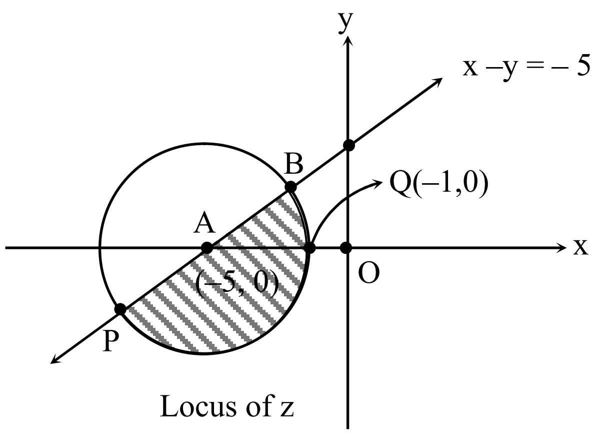
(iv) + (v) $\Rightarrow$ b = 0 a = 3

$\therefore\ a^2 + b^2 + ab = 9$

24. Let z(z ∈ C) satisfy $|z + 5| \leq 5$ and $z(1 + i) + \bar{z}(1 - i) \geq -10$ if the maximum value of $|z + 1|^2$ is $\alpha + \beta\sqrt{2}$; then find $\alpha + \beta$

Ans. 48

Sol.



$\therefore$ P ≡ $(-5 - 2\sqrt{2}, -2\sqrt{2})$

$\therefore$ $(PQ)^2\big|_{max} = 32 + 16\sqrt{2}$

$\alpha = 32$

$\beta = 16$

$\alpha + \beta = 48$

25. –16, 8, –4, 2 ...... is sequence, whose AM & GM of p^{th} & q^{th} term are the roots of $4x^2 - 9x + 5 = 0$, Find p + q

Ans. 10

Sol. –16, 8, –4, 2 ......

p^{th} term $t_p = -16\left(\frac{-1}{2}\right)^{p-1}$

q^{th} term $t_q = -16\left(\frac{-1}{2}\right)^{q-1}$

Now $\frac{t_p + t_q}{2} = \frac{5}{4} \& \sqrt{t_p t_q} = 1$

$\Rightarrow 16^2\left(-\frac{1}{2}\right)^{p+q-2} = 1$

$\Rightarrow 2^8 = (-2)^{(p+q-2)}$

$\Rightarrow p + q = 10$

26. If x_1, x_2, x_3 ……x_{18} are 18 terms and $\sum_{1}^{18}(x_i - \alpha) = 36$, $\sum_{1}^{18}(x_i - \beta)^2 = 90$ variance $(\sigma^2) = 1$ given, then find $|\beta - \alpha|$

Ans. 4

Sol. $\Sigma x_i - 18\alpha = 36$

$\Sigma x_i = 18(\alpha + 2)$ …(i)

$\Sigma x_i^2 + 18\beta^2 - 2\beta\, \Sigma x_i = 90$

$\Sigma x_i^2 + 18\beta^2 - 2\beta \times 18(\alpha + 2) = 90$

$\Sigma x_i^2 = 90 - 18\beta^2 + 36\beta(\alpha + 2)$ …(ii)

$\sigma^2 = 1 \Rightarrow \frac{1}{18}\Sigma x_i^2 - \left(\frac{\Sigma x_i}{18}\right)^2 = 1$

$\Rightarrow \frac{1}{18}(90 - 18\beta^2 + 36\alpha\beta + 72\beta) - \left(\frac{18(\alpha+2)}{18}\right)^2 = 1$

$\Rightarrow 90 - 18\beta^2 + 36\alpha\beta + 72\beta - 18(\alpha + 2)^2 = 18$

$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - (\alpha + 2)^2 = 1$

$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - \alpha^2 - 4 - 4\alpha = 1$

$-\alpha^2 - \beta^2 + 2\alpha\beta + 4\beta - 4\alpha = 0$

$- \quad (\alpha - \beta)^2 - 4(\alpha - \beta) = 0$

$-(\alpha - \beta)(\alpha - \beta + 4) = 0$

$\Rightarrow \alpha - \beta = -4 \qquad\qquad (\alpha \neq \beta)$

$|\beta - \alpha| = 4$

27. It $P_n = \alpha^n + \beta^n$, $\alpha + \beta = 1$, $\alpha\beta = -1$, $P_{n-1} = 11$, $P_{n+1} = 29$ find $(P_n)^2$ (where $n \in N$)

Ans. 324

Sol. Quadratic Equation whose roots are α, β : $x^2 - x - 1 = 0$

$\therefore \alpha^2 = \alpha + 1 \Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2}$

$\beta^2 = \beta + 1 \Rightarrow \beta^n = \beta^{n-1} + \beta^{n-2}$

$\therefore P_n = P_{n-1} + P_{n-2}$

$\Rightarrow P_{n+1} = P_n + P_{n-1}$

$\Rightarrow 29 = P_n + 11 \Rightarrow P_n = 18$

$\Rightarrow (Pn)^2 = 324$

28. The number of four digit numbers whose HCF with 18 is 3 equals

Ans. 1000

Sol. Number must be an odd multiple of 3 and not a multiple of 9

4-digit odd multiples of 3 are

1005, 1011, ……., 9999 $\rightarrow$ 1499

4-digit odd multiples of 9 are

1017, 1035, ……., 9999 $\rightarrow$ 499

$\therefore$ Required numbers $\rightarrow$ 1000

29. Image of a point (1, 0, –1) in the plane $4x - 5y + 2z = 8$ is (α, β, γ). Find $15(\alpha + \beta + \gamma)$

Ans. 4

Sol. $\dfrac{x-1}{4} = \dfrac{y-0}{-5} = \dfrac{z+1}{2} = \dfrac{-2(-6)}{16+25+4} = \dfrac{12}{45} = \dfrac{4}{15}$

$x - 1 = \dfrac{16}{15} \Rightarrow x = \dfrac{31}{15}$

$y = -\dfrac{4}{3}$

$z + 1 = \dfrac{8}{15} \Rightarrow z = -\dfrac{7}{15}$

$\alpha = \dfrac{31}{15}, \beta = -\dfrac{4}{3}, \gamma = -\dfrac{7}{15}$

$15(\alpha + \beta + \gamma) = \left(\dfrac{31}{15} - \dfrac{4}{3} - \dfrac{7}{15}\right) \times 15 = 4$

30. If $f(x) = 2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$ and all real roots of f(x) lie in the interval $(-\alpha, -\alpha+1)$ then 'α' is :

Ans. 2

Sol. $f(-1) = 3 > 0$

$f(-2) = -64 + 80 - 80 + 40 - 20 + 10$

$= -34 < 0$

$\therefore$ At least one root in $(-2,-1)$

$f'(x) = 10\,(x^4 + 2x^3 + 3x^2 + 2x + 1)$

$$= 10\left(x^2 + \frac{1}{x^2} + 2\left(x + \frac{1}{x}\right) + 3\right)$$

$$= 10\left(\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) + 1\right)$$

$$= 10\left(\left(x + \frac{1}{x}\right) + 1\right)^2 > 0; \forall x \in R$$

$\therefore$ Exactly one real root in $(-2,-1)$